Integration Constants as State Variables for Optimal Path Planning

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***Abstract*— This paper explores the advantages of a state parameterization in integration constants to optimally path plan multi-dimensional systems governed by ordinary differential equations. The idea is inherited from astrodynamics, where the integration constants of the unperturbed equations of orbital motion – orbital elements – are used to path plan optimally artificial satellites in space. This paper shows that the advantages gained from this state parameterization can be extended to other applications. In order to do so, firstly, it presents a general methodology to model the governing equations – in general non-integrable and nonlinear – using the integration constants of a reduced integrable part. Subsequently, it formalizes the optimal path planning problem in integration constant space, where a geometric interpretation of the minimum input cost path is provided together with an algorithmic procedure to identify it. It is shown that working in integration constant space simplifies the path planning. In particular, the integrable dynamics can be exploited intrinsically, while the part of the dynamics which is not integrable can be isolated and compensated.**

I. INTRODUCTION

The use of integration constants (IC) as state variables is inherited from astrodynamics. The well known Keplerian orbital elements are IC of the homogeneous orbital ordinary differential equations (ODE) of motion [1]. Orbital elements are currently broadly used on ground and in-flight for all kind of applications, from dynamic modeling and orbit design to spacecraft guidance navigation and control (GN&C). In addition, more recently, orbital elements have been employed to model the relative position of multi-agents space systems by means of the definition of the relative orbital elements (ROE) state [2][3]. A recent finding about the use of ROE has attracted the attention for its simplicity and efficacy. In fact, the problem of optimally guide and control the relative position of two satellites in space (or one satellite with respect to a reference orbit) can be solved by a geometrical approach in the ROE space [2][4]. This discovery permitted to optimally and autonomously path-plan the satellite relative trajectories in two space missions in Low Earth Orbit: TanDEM-X [6] and PRISMA [7].

The simplicity of the developed algorithms motivated the following question: can the advantages of an IC parameterization of the state be generalized to the optimal path planning of any multi-dimensional systems governed by ODE? This paper shows that the answer is yes, and presents a mathematical demonstration and application methodology. Firstly, in Sec.II-III, the paper presents how the governing ODE – in general non-integrable and nonlinear – can be modeled using a state composed by the IC of a reduced integrable part. The proposed modeling approach is inspired by the well known Variation of Parameters method [8][9]. Secondly, in Sec.IV, the paper defines the minimum control input path planning problem in IC space and, leveraging relevant results from reachable set theory and optimal control theory [10][11][12][13], provides a geometric interpretation and a resolution algorithm to the problem. Finally, in Sec.V, the paper shows how the theoretical results apply to two very different problems involving multi-dimensional systems governed by ODE: the path planning of an unmanned aerial vehicle (UAV), and the control of the relative motion between spacecraft. The two examples demonstrate the consistency of the proposed approach and the simplification that working in IC space introduces to the optimal path planning.

II. MODELING APPROACH USING INTEGRATION CONSTANTS

Let us consider the generic ODE of n^{th} order

$$\chi^{(n)} = F(t,\boldsymbol{\chi}) = F(t,\chi,\dot{\chi},\ddot{\chi},...,\chi^{(n-1)}) \tag{1}$$

where t is the independent variable (in the following of the paper identified with time) and $\chi(t)$ is the dependent operative variable. Eq.1 can be always reduced to a 1^{st} order system of n-ODE

$$\dot{\boldsymbol{\chi}} = \boldsymbol{F}(t,\boldsymbol{\chi}) \tag{2}$$

where, $\boldsymbol{\chi} = [\chi,\dot{\chi},\ddot{\chi},...,\chi^{(n-1)}]^T$ and $\boldsymbol{F}(t,\boldsymbol{\chi}) = [\dot{\chi},\ddot{\chi},...,\chi^{(n-1)},F(t,\boldsymbol{\chi})]^T$. Here, the exponent T means transpose. If Eq. 1 is integrable, a closed form map h exists, such that

$$\boldsymbol{\chi}(t) = h(t,t_0,\boldsymbol{c}) \qquad \boldsymbol{c} = h^{-1}(t,t_0,\boldsymbol{\chi}(t)) \tag{3}$$

where $\boldsymbol{c} = [c_1,...,c_n]$ is the integration constants (IC) state. The IC state elements are independent linear/nonlinear combination of the operative variable initial conditions $\boldsymbol{\chi}(t_0) = [\chi_0,\dot{\chi}_0,...,\chi_0^{(n-1)}]^T$ and may depend on the selected initial integration instant t_0. In literature, other definitions of integration/motion constants (or integrals) can be found [14][15]. In particular, the distinction between $n-1$ time-independent integrals and one t_0 dependent integral is rigorously made [15]. In this paper, the IC are the n constants associated to the operative state initial condition $\boldsymbol{\chi}(t_0)$ and t_0. Finally, the map $h(t,t_0,\boldsymbol{\chi}(t))$ is assumed here to be continuous, differentiable, invertible, and possible ambiguities for multivalued functions can be discriminated.

Given this general mathematical framework, the proposed

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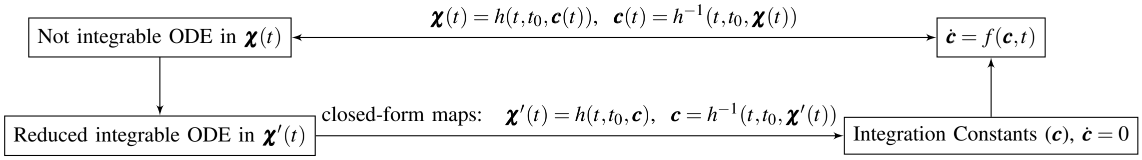

Fig. 1. Visualization of the proposed approach. The original non-integrable multi-dimensional system of ODE (top left) is reduced to a simpler integrable form in the operative state variables (bottom left). A closed form map allows to recast the reduced system in IC-state space (bottom right). A new set of first order differential equations govern the evolution of the new IC state parameters (top right).

modeling approach using IC is described in Fig. 1. Four key steps are defined: 1) the dynamics of the modeled system is, in general, described by an n^{th} order non-integrable ODE in the operative state variables, which can be reduced to a 1^{st} order system of n-ODE (as shown in Eq.2); 2) the non-integrable ODE can be expressed, without altering the nature of the operative state, as the superposition of an integrable (reduced) part and the remaining (perturbative) part which makes it non-integrable

$$\dot{\boldsymbol{\chi}} = \boldsymbol{F}(t,\boldsymbol{\chi}) = \boldsymbol{F}'(t,\boldsymbol{\chi}) + \boldsymbol{F}_p(t,\boldsymbol{\chi},\boldsymbol{\beta}) \tag{4}$$

The perturbative part, $\boldsymbol{F}_p$, contains all the terms of the governing equations which make them not integrable and is in general a function of both the operative variables, $\boldsymbol{\chi}$, and other parameters, $\boldsymbol{\beta}$, omitted in the following notation for compactness. The term $\boldsymbol{F}_p(t,\boldsymbol{\chi})$ can, for example, include nonlinear parts of the governing equations which make them non integrable or randomly generated perturbations related to unmodeled disturbances. As it will be evident in the following of the paper, the more dynamics is included in the reduced integrable part, the more the proposed approach carries advantage; 3) the reduced integrable ODE is integrated and the IC state is defined together with the closed form map which links it to the operative state: $\dot{\boldsymbol{\chi}}' = \boldsymbol{F}'(t,\boldsymbol{\chi}') \leftrightarrow \boldsymbol{\chi}'(t) = h(t,t_0,\boldsymbol{c})$. Finally, 4) a set of ODE of IC variation need to be derived to model the original non-integrable operative system. These equations, called from here on ICV equations, are formalized in the following section.

III. INTEGRATION CONSTANTS VARIATION EQUATION

The following steps are inspired by the well known Variation of Parameters (VOP) method [8][9], and show the general mathematical formulation of the ICV equations. Eq.4 can be expanded as

$$\dot{\boldsymbol{\chi}}(t,\boldsymbol{c}) = \frac{\partial \boldsymbol{\chi}(t,\boldsymbol{c})}{\partial t} + \sum_{k=1}^{n} \frac{\partial \boldsymbol{\chi}(t,\boldsymbol{c})}{\partial c_k}\frac{\partial c_k}{\partial t} = \boldsymbol{F}'(t,\boldsymbol{\chi}) + \boldsymbol{F}_p(t,\boldsymbol{\chi}) \tag{5}$$

where c_k is the k^{th} component ($k = 1,...,n$) of the IC state. Eq.5 represents the fact that the variation in time of the operative state governed by the non-integrable dynamics can be attributed to a component independent from the IC variation and another component related to the IC variation. By definition, the IC state is constant under the reduced integrable dynamics, therefore the following fundamental condition holds

$$\frac{\partial \boldsymbol{\chi}}{\partial t} = \frac{\partial \boldsymbol{\chi}'}{\partial t} = \boldsymbol{F}'(t,\boldsymbol{\chi}) \tag{6}$$

Eq.6 can be defined as the generalized form of the *osculation condition*, which was firstly introduced by Lagrange to derive the VOP [8][3]. The osculation condition represents the fact that at each instant of time, fixed the instantaneous value of IC, the trajectory given by the non-integrable dynamics is tangent to the one given by the reduced integrable dynamics. Combining Eq.5 and Eq.6, it is possible to write in vectorial form

$$\frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{c}}(t,\boldsymbol{c})\dot{\boldsymbol{c}} = \boldsymbol{F}_p(t,\boldsymbol{\chi}) \tag{7}$$

where $\frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{c}}(t,\boldsymbol{c})$ is the Jacobian matrix of the closed form map $\boldsymbol{\chi}(t) = h(t,t_0,\boldsymbol{c})$ and is supposed to exist continuous and invertible for $t > t_0$. This Jacobian is a function of time and, in the general case, it may also be a function of the IC. The possible dependence of the Jacobian from the IC is a key aspect, as presented in the following of the section. Inverting Eq.7, the Integration Constants Variation (ICV) equation is formalized as

$$\dot{\boldsymbol{c}} = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}}(t,\boldsymbol{c})\boldsymbol{F}_p(t,\boldsymbol{\chi}) \tag{8}$$

Therefore, the original non-integrable ODE in the operative variables (Eq.4) is equivalent to the ICV equation (Eq.8), in which the IC of the reduced integrable ODE vary in time under the pure action of the perturbative term.

A. Application to Translational Motion of a Mass Particle

As mentioned, the ICV equation takes direct inspiration from the well known VOP method [8][9]. The relation between ICV equation and VOP is formalized in this paragraph in which the methodology presented in Fig.1 is applied to the translational motion of a mass particle. In this case, the operative space is the Cartesian space and the original non-integrable dynamics is represented by the 2^{nd} order ODE given by application of the second Newton law: $\ddot{\boldsymbol{r}} = \boldsymbol{f}(t,\boldsymbol{r},\dot{\boldsymbol{r}}) = \boldsymbol{f}'(t,\boldsymbol{r},\dot{\boldsymbol{r}}) + \boldsymbol{f}_p(t,\boldsymbol{r},\dot{\boldsymbol{r}})$, where $\boldsymbol{r}$,$\dot{\boldsymbol{r}}$ are the Cartesian position and velocity vectors and the terms $\boldsymbol{f}$ represent acceleration vectors. Following the presented methodology, the non integrable dynamics has been expressed as superposition of a reduced integrable and a perturbative part. The reduced integrable part can be integrated and the IC state, $\boldsymbol{c}$, can be defined: $\ddot{\boldsymbol{r}}' = \boldsymbol{f}'(t,\boldsymbol{r}',\dot{\boldsymbol{r}}') \leftrightarrow [\boldsymbol{r}'(t),\dot{\boldsymbol{r}}'(t)] = h(t,t_0,\boldsymbol{c})$. Moreover, the 2^{nd} order ODE can be expressed as a system of two-first order ODE by state augmentation: $\boldsymbol{\chi} = [\boldsymbol{r},\dot{\boldsymbol{r}}]^T$,

$\boldsymbol{F} = [\dot{\boldsymbol{r}}, \boldsymbol{f}]^T$. Subsequently, the ICV equation can be applied

$$\dot{\boldsymbol{c}} = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}} \boldsymbol{F}_p(t,\boldsymbol{\chi}) = \frac{\partial \boldsymbol{c}}{\partial \dot{\boldsymbol{r}}} \boldsymbol{f}_p(t,\boldsymbol{\chi}) = \boldsymbol{P}^T \left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{c}}\right)^T \boldsymbol{f}_p(t,\boldsymbol{\chi}) \quad (9)$$

The second term in the equality is the Gauss VOP [1], the third term in the equality is the Lagrange VOP [1][3] with $\boldsymbol{f}_p = -\partial V/\partial \boldsymbol{r}$ (where V is a potential function), the matrix $\boldsymbol{P} = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{r}} \left(\frac{\partial \boldsymbol{c}}{\partial \dot{\boldsymbol{r}}}\right)^T - \frac{\partial \boldsymbol{c}}{\partial \dot{\boldsymbol{r}}} \left(\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{r}}\right)^T$ is the Poisson Matrix [3][14].

B. Control Action Introduction

In the most general case, the control action $\boldsymbol{u}(t) = [u_1(t), ..., u_m(t)]^T$ can be introduced in Eq.4 as $\dot{\boldsymbol{\chi}} = \boldsymbol{F}(t,\boldsymbol{\chi},\boldsymbol{u})$. In this paper, it is considered only the case in which the control action enters linearly into the governing ODE as

$$\dot{\boldsymbol{\chi}} = \boldsymbol{F}(t,\boldsymbol{\chi}) + \boldsymbol{B}(t)\boldsymbol{u}(t) \quad (10)$$

which implies that terms $\mathcal{O}(\boldsymbol{u}^2, \boldsymbol{u}\boldsymbol{\chi})$ are small and neglected. $\boldsymbol{B}(t)$ is the control input matrix of dimensions $[n \times m]$. Given these assumptions, the control enters in Eq.8 as

$$\dot{\boldsymbol{c}} = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}}(t,\boldsymbol{c}) \boldsymbol{F}_p(t,\boldsymbol{\chi}) + \boldsymbol{\Gamma}(t,\boldsymbol{c})\boldsymbol{u}(t) \quad (11)$$

where the control input matrix in given by $\boldsymbol{\Gamma}(t,\boldsymbol{c}) = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}}(t,\boldsymbol{c})\boldsymbol{B}(t)$. The way control action is inserted directly on the IC in Eq.11 constitutes a generalization of what is done for spacecraft control using Gauss VOP [1][2][3].

C. Structure of the Jacobian Matrix

The structure of the Jacobian matrix – $\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}}(t,\boldsymbol{c})$ – influences how the perturbing part of the dynamics and the control affect the evolution in time of the IC state. The Jacobian structure is determined by: 1) the choice of the reduced integrable dynamics, 2) the choice of the IC state. These two choices can be made to provide the Jacobian with a favorable structure. Starting form choice 1), if the reduced integrable dynamics is selected to be a *linear* ODE of the form $\dot{\boldsymbol{\chi}}' = \boldsymbol{A}(t)\boldsymbol{\chi}'$, the map

$$\boldsymbol{\chi}' = h(t,t_0,\boldsymbol{c}) = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{c}}(t,t_0)\boldsymbol{c} = \boldsymbol{\Psi}(t,t_0)\boldsymbol{c} \quad (12)$$

is linear and its Jacobian coincides with the fundamental matrix – $\boldsymbol{\Psi}(t,t_0)$ – which links IC and operative states and depends on time but not on the IC themselves. In particular, each column of $\boldsymbol{\Psi}(t,t_0)$ is the solution of the linear ODE [9](p.2). *The selection of a linear reduced integrable dynamics does not reduce the generality of the approach.* In fact, the variation with respect to a reference operative state can be considered: $\delta\boldsymbol{\chi} = \boldsymbol{\chi} - \boldsymbol{\chi}_{ref}$, and Eq.10 can be Taylor expanded in its neighborhood such as

$$\delta\dot{\boldsymbol{\chi}} = \dot{\boldsymbol{\chi}} - \dot{\boldsymbol{\chi}}_{ref} = \frac{\partial \dot{\boldsymbol{\chi}}}{\partial \boldsymbol{\chi}}\Big|_{\boldsymbol{\chi}=\boldsymbol{\chi}_{ref}} \delta\boldsymbol{\chi} + \boldsymbol{B}\boldsymbol{u} + \mathcal{O}(\delta\boldsymbol{\chi}^2) \quad (13)$$

Given this expansion, the reduced integrable and perturbative dynamics can be selected such that $\boldsymbol{F}'(t,\boldsymbol{\chi}) = \frac{\partial \dot{\boldsymbol{\chi}}}{\partial \boldsymbol{\chi}_{ref}}\Big|_{\boldsymbol{\chi}=\boldsymbol{\chi}_{ref}} \delta\boldsymbol{\chi}$ is a linear ODE and $\boldsymbol{F}_p(t,\boldsymbol{\chi}) = \mathcal{O}(\delta\boldsymbol{\chi}^2)$ contains all the higher order terms. The linear reduced integrable dynamics leads to an IC-independent Jacobian (Eq.12) and so to an IC-independent control input matrix. In particular, Eq.11 becomes

$$\dot{\boldsymbol{c}} = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}}(t) \boldsymbol{F}_p(t,\boldsymbol{\chi}) + \boldsymbol{\Gamma}(t)\boldsymbol{u}(t) \quad (14)$$

Eq.14 is *exactly* equivalent to Eq.10-11 (if $\dot{\boldsymbol{\chi}}_{ref} = 0$) and (in general) to Eq.13, since all the higher order terms are contained in $\boldsymbol{F}_p$. Moreover, in Eq.14, the relation between IC state and control is Linear Time Variant (LTV), a favorable propriety from a control perspective. Regarding choice 2), the IC state can be selected to provide specific properties to the Jacobian and so to the control input matrix. For example, decoupling specific IC state components with respect to specific control input components, may permit to solve the full ($n-$dimensional) control problem as the union of lower dimensional control problems, fostering geometric approaches.

IV. MINIMUM CONTROL INPUT PROBLEM IN INTEGRATION CONSTANT SPACE

This section shows the advantages of an IC parameterization of the state. In IC space, the integrable part of the dynamics is intrinsically leveraged in the solution of the optimal control problem. The perturbative part is directly isolated and then completely, or at least partially, compensated. In addition, this section provides a geometric interpretation of minimum input cost paths in IC space, opening the way to a possible direct geometrical resolution of the optimal control problem. The working assumptions are:

1) The four steps in Fig.1 have been applied to the non-integrable ODE (Eq.10) governing the operative state, leading to the definition of the IC state, and of the closed form map which links it to the operative state. The closed form map is continuous, differentiable, invertible and possible multivalued ambiguities can be discriminated. Its Jacobian matrix exists continuous and invertible for $t > t_0$.
2) The dynamics governing the IC state is modeled by Eq.14, with Jacobian and control input matrices IC-independent. This, as explained in Sec.III-C, does not introduce mathematically any approximation on the original system modeled by Eq.10.
3) The admissible control action $\boldsymbol{u}(t)$ is modeled on the control interval $[t_0, t_f]$ as a discrete sequence of impulses applied at instants $t_i \in [t_0, t_f]$, such that $\boldsymbol{u}(t) = \sum_{i=1}^{N} \Delta\boldsymbol{u}(t_i)$, with N finite integer.

Under these assumptions, the optimal path planning problem in operative space can be formalized as

$$\begin{aligned} \underset{\boldsymbol{u}}{\text{minimize}} \quad & J = \sum_{i=1}^{N} ||\Delta\boldsymbol{u}(t_i)|| \\ \text{subject to} \quad & \dot{\boldsymbol{\chi}} = \boldsymbol{F}'(t,\boldsymbol{\chi}) + \boldsymbol{F}_p(t,\boldsymbol{\chi}) + \sum_{i=1}^{N} \boldsymbol{B}(t_i)\Delta\boldsymbol{u}(t_i) \\ & \boldsymbol{\chi}(t_0) = \boldsymbol{\chi}_0 \quad \boldsymbol{\chi}(t_f) = \boldsymbol{\chi}_{goal} \end{aligned} \quad (15)$$

where $||.||$ represents the $2-$norm. According to assumption 2), this optimization problem can be expressed equivalently in IC space as

$$\begin{aligned} \underset{\boldsymbol{u}}{\text{minimize}} \quad & J = \sum_{i=1}^{N} ||\Delta \boldsymbol{u}(t_i)|| \\ \text{subject to} \quad & \dot{\boldsymbol{c}} = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}}(t) \boldsymbol{F}_p(t,\boldsymbol{c}) + \sum_{i=1}^{N} \boldsymbol{\Gamma}(t_i)\Delta \boldsymbol{u}(t_i) \\ & \boldsymbol{c}(t_0) = \boldsymbol{c}_0 \quad \boldsymbol{c}(t_f) = \boldsymbol{c}_{goal} \end{aligned} \tag{16}$$

where the initial and target IC states are obtained through closed form map: $\boldsymbol{c}_0 = h^{-1}(t_0, t_0, \boldsymbol{\chi}_0)$, $\boldsymbol{c}_{goal} = h^{-1}(t_f, t_0, \boldsymbol{\chi}_{goal})$, and $\boldsymbol{F}_p$ is expressed as function of the IC state. Being equivalent, Eq.15 and Eq.16 necessarily lead to the same optimal control solution. On the other hand, while the operative state varies under superposition of integrable dynamics, $\boldsymbol{F}_p$ and control action, the IC vary affected by only the latter. By starting neglecting $\boldsymbol{F}_p$, a geometric interpretation of the minimum input cost path in IC-space is given in Sec.IV-A, and it is visualized in Sec.V-A. Then, two possible ways of leveraging $\boldsymbol{F}_p$ are proposed in Sec.IV-C, and they are contextualized in Sec.V-B.

A. Reachable IC Set and Minimum Cost Paths in IC Space

According to assumption 3) and neglecting $\boldsymbol{F}_p$, the link between IC state variation and control action is: $\Delta \boldsymbol{c} = \sum_{i=1}^{N} \Delta \boldsymbol{c}(t_i) = \sum_{i=1}^{N} \boldsymbol{\Gamma}(t_i)\Delta \boldsymbol{u}(t_i)$. The set of control inputs with cost lower equal than J is defined as

$$\mathscr{U}(J) = \big\{ \boldsymbol{u}(t) : \boldsymbol{u}(t) = \sum_{i=1}^{N} \Delta \boldsymbol{u}(t_i),\ \sum_{i=1}^{N} ||\Delta \boldsymbol{u}(t_i)|| \leq J \big\} \tag{17}$$

which is a compact and convex set $\forall N \geq 1$, representing a hypersphere of radius J. The corresponding set of reachable IC variations is

$$\mathscr{C}(J) = \big\{ \Delta \boldsymbol{c} : \Delta \boldsymbol{c} = \sum_{i=1}^{N} \boldsymbol{\Gamma}(t_i)\Delta \boldsymbol{u}(t_i),\ \sum_{i=1}^{N} ||\Delta \boldsymbol{u}(t_i)|| \leq J \big\} \tag{18}$$

$\mathscr{C}$ is compact, convex and scales linearly with J, the proof of these statements for generic LTV system is given in [11](p.3, *Theorem 1*) and [10](p.326, *Lemma*).
If the variation $\Delta \boldsymbol{c}_{goal} = \boldsymbol{c}_{goal} - \boldsymbol{c}_0$ is reachable by a series of N admissible impulses, $\Delta \boldsymbol{c}_{goal}$ must lie on the border (∂.) of the set $\mathscr{C}(J_{min}) = \big\{ \Delta \boldsymbol{c} : \Delta \boldsymbol{c} = \sum_{i=1}^{N} \boldsymbol{\Gamma}(t_i)\Delta \boldsymbol{u}(t_i),\ \sum_{i=1}^{N} ||\Delta \boldsymbol{u}(t_i)|| \leq J_{min} \big\}$, where J_{min} is the optimal minimum cost necessary to achieve the desired IC variation. Given this evidence, [11](p.4, *Theorem 2*) and [10](p.326, *Corollary*) report a fundamental result that, in the following, is applied in IC space to the problem of moving the state optimally from $\boldsymbol{c}_0$ to $\boldsymbol{c}_{goal}$. Let $\hat{\boldsymbol{\eta}}$ be an exterior normal unit vector to the convex set $\mathscr{C}(J_{min})$ at $\Delta \boldsymbol{c}_{goal}$, the halfspace $\{\Delta \boldsymbol{c} : \hat{\boldsymbol{\eta}}^T \Delta \boldsymbol{c} \leq \hat{\boldsymbol{\eta}}^T \Delta \boldsymbol{c}_{goal}\}$ contains $\mathscr{C}(J_{min})$ [13](p.50-51), then

$$\begin{aligned} \hat{\boldsymbol{\eta}}^T \Delta \boldsymbol{c}_{goal} &= \max_{\Delta \boldsymbol{c} \in \mathscr{C}(J_{min})} \hat{\boldsymbol{\eta}}^T \Delta \boldsymbol{c} = \max_{\boldsymbol{u}(t) \in \mathscr{U}(J_{min})} \hat{\boldsymbol{\eta}}^T \big[\sum_{i=1}^{N} \boldsymbol{\Gamma}(t_i)\Delta \boldsymbol{u}(t_i) \big] \\ &= J_{min} \max_{\boldsymbol{u}(t) \in \mathscr{U}(1)} \big[\sum_{i=1}^{N} \hat{\boldsymbol{\eta}}^T \boldsymbol{\Gamma}(t_i)\Delta \boldsymbol{u}(t_i) \big] = J_{min} \max_{t_i \in [t_0, t_f]} ||\hat{\boldsymbol{\eta}}^T \boldsymbol{\Gamma}(t_i)|| \end{aligned} \tag{19}$$

where the last relation holds for the scenario of $\boldsymbol{u}(t) \in \partial \mathscr{U}(1)$ (i.e. $\sum_{i=1}^{N} ||\Delta \boldsymbol{u}(t_i)|| = 1$) with each $\Delta \boldsymbol{u}(t_i)$ applied parallel to the vector $\hat{\boldsymbol{\eta}}^T \boldsymbol{\Gamma}(t_i)$ when $||\hat{\boldsymbol{\eta}}^T \boldsymbol{\Gamma}(t_i)||$ is maximal. This leads to the fundamental result

$$J_{min} = \frac{\hat{\boldsymbol{\eta}}^T \Delta \boldsymbol{c}_{goal}}{\max_{t_i \in [t_0, t_f]} ||\hat{\boldsymbol{\eta}}^T \boldsymbol{\Gamma}(t_i)||} \tag{20}$$

consistent with [11] and [10]. Eq.20 states that the minimum cost required to achieve $\Delta \boldsymbol{c}_{goal}$ is dictated by its scaled projection on the unit vector $\hat{\boldsymbol{\eta}}$ (which defines the direction normal to $\partial \mathscr{C}(J_{min})$ at $\Delta \boldsymbol{c}_{goal}$). This scaled projection is by definition proportional to $||\Delta \boldsymbol{c}_{goal}||$, in accordance to the fact that the set $\mathscr{C}(J)$ scales linearly with J. In the limiting case of $\Delta \boldsymbol{c}_{goal}$ and $\hat{\boldsymbol{\eta}}$ *parallel*, the minimum cost is uniquely dictated by the Euclidean distance from $\boldsymbol{c}_0$ to $\boldsymbol{c}_{goal}$. This case is realized if $\Delta \boldsymbol{c}_{goal}$ is perpendicular to $\partial \mathscr{C}(J_{min})$. Furthermore, Eq.20 implies that all optimal feasible paths from $\boldsymbol{c}_0$ to $\boldsymbol{c}_{goal}$ must share the projection on $\hat{\boldsymbol{\eta}}$, or in other words, *the component of the IC motion that is paid in an optimal transfer from initial to target state is the one along* $\hat{\boldsymbol{\eta}}$ *direction*. This is a geometric criteria which characterizes minimum cost paths. Given this criteria, a minimum length path is not necessary an optimal solution. It is optimal only if: $\exists t_i \in [t_0, t_f]$ such that $\Delta \boldsymbol{c}_{goal} = \boldsymbol{\Gamma}(t_i)\Delta \boldsymbol{u}(t_i)$ and $||\Delta \boldsymbol{u}(t_i)|| = J_{min}$, i.e., the required IC variation can be acquired by a single minimum cost admissible control impulse over the control interval. An exception is the case in which $\boldsymbol{\Gamma}(t)$ is periodic over the control interval, so $\exists N$ instants $t_i \in [t_0, t_f]$ such that $\boldsymbol{\Gamma}(t_1) = .. = \boldsymbol{\Gamma}(t_i) = .. = \boldsymbol{\Gamma}(t_N)$, and $\Delta \boldsymbol{c}_{goal} = \sum_{i=1}^{N} \boldsymbol{\Gamma}(t_i)\Delta \boldsymbol{u}(t_i)$, with $\sum_{i=1}^{N} ||\Delta \boldsymbol{u}(t_i)|| = J_{min}$. Besides the previous cases, a minimum length path in IC space may constitute a sub-optimal solution. The degree of sub-optimality depends on the features of the reachable IC set over the selected control interval. These are dictated by the structure of the control input and Jacobian matrices, and so by the selection of IC state and reduced integrable dynamics. In addition, practically, in common path planning application scenarios a single impulsive solution may not be sufficient to achieve the desired state reconfiguration. An example is given in Sec.V-A.
It is relevant to notice that the result presented in Eq.20 is general for LTV systems and can be applied also in the operative space. Nevertheless, the peculiarity of the IC space (neglecting $\boldsymbol{F}_p$) is that the optimal path is exclusively composed by state variations due to control input. In operative space instead, the controlled path is superposed with the free-motion due to the integrable dynamics. This prevents the direct geometric identification of the optimal path, which is possible instead in IC space (see Sec.V-A) and may open the way to a geometric solution of the optimal control problem.

B. Dual Optimization Problem and Solution Algorithm

Sec.IV-A highlights the importance of the unit vector $\hat{\boldsymbol{\eta}}$, which defines a fundamental direction in IC space. Following [13](p.215), it is possible to identify $\hat{\boldsymbol{\eta}}$ as the unit vector of the Lagrange multiplier ($\boldsymbol{\eta}$) associated with the equality constraint in Eq.16. $\boldsymbol{\eta}$ is therefore a dual variable associated with the problem in Eq.16, and its optimal value identifies the direction – $\hat{\boldsymbol{\eta}} = \boldsymbol{\eta}_{opt.}/||\boldsymbol{\eta}_{opt.}||$ – exterior normal to the convex set $\mathscr{C}(J_{min})$. This optimal value can be found by solving the dual problem

$$\begin{aligned} &\underset{\boldsymbol{\eta}}{\text{maximize}} \quad J_{dual} = \boldsymbol{\eta}^T \Delta \boldsymbol{c}_{goal} \\ &\text{subject to} \quad ||\boldsymbol{\Gamma}(t_j)^T \boldsymbol{\eta}|| \leq 1 \quad \text{for} \quad t_j \in [t_0, t_f] \end{aligned} \tag{21}$$

where $[t_0, ..t_j, ..t_f]$ is a defined discretization of the control interval. Inspired by [12](p.377, *Algorithm 1*), [11] and [10](p.328, *Theorem 3*), the following algorithm is proposed to identify the associated optimal control action

1) solve problem in Eq.21 for $\boldsymbol{\eta}_{opt.}$.
2) find $t_i \in [t_0, t_f]$ such that $||\boldsymbol{\Gamma}(t_i)^T \boldsymbol{\eta}_{opt.}|| = 1$.
3) the i^{th} impulse is directed as $\Delta\hat{\boldsymbol{u}}(t_i) = \boldsymbol{\Gamma}(t_i)^T \boldsymbol{\eta}_{opt.}$.
4) the impulses norms have to satisfy the linear system of N unknowns $||\Delta\boldsymbol{u}(t_i)||$ in n equations:

$$\begin{bmatrix} \boldsymbol{\Gamma}(t_1)\Delta\hat{\boldsymbol{u}}(t_1) & .. & \boldsymbol{\Gamma}(t_i)\Delta\hat{\boldsymbol{u}}(t_i) & .. & \boldsymbol{\Gamma}(t_N)\Delta\hat{\boldsymbol{u}}(t_N) \end{bmatrix} \begin{bmatrix} ||\Delta\boldsymbol{u}(t_1)|| \\ .. \\ ||\Delta\boldsymbol{u}(t_i)|| \\ .. \\ ||\Delta\boldsymbol{u}(t_N)|| \end{bmatrix} = \Delta\boldsymbol{c}_{goal} \tag{22}$$

The maximum number of impulses necessary does not exceed the dimensions of the IC state ($N \leq n$), [11].

C. Perturbative Dynamics Compensation

In the complete system, the perturbative term – $\boldsymbol{F}_p$ – and the control action act simultaneous on the IC state. The effect of $\boldsymbol{F}_p$ has to be leveraged, if possible, in order to reduce $||\Delta\boldsymbol{c}_{goal}||$ and so the control cost, which scales linearly with it. Two cases of $\boldsymbol{F}_p$ are analyzed in the following.

1) IC-independent $\boldsymbol{F}_p$: in this case, control action and $\boldsymbol{F}_p$ are uncoupled on the interval $[t_0, t_f]$. The integrated effect of $\boldsymbol{F}_p$ on the interval can be computed and used to compensate the local IC target state as: $\boldsymbol{c}^*_{goal} = \boldsymbol{c}_{goal} - \int_{t_0}^{t_f} \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}} \boldsymbol{F}_p dt$. Where, $\boldsymbol{c}^*_{goal}$ is the compensated target state and $\boldsymbol{F}_p$ is supposed known and modeled.

2) IC-dependent $\boldsymbol{F}_p$: in this case, control action and $\boldsymbol{F}_p$ are coupled. A possible solution is to define a relative IC state with respect to a reference as: $\boldsymbol{\delta c} = \boldsymbol{c} - \boldsymbol{c}_{ref}$. Then, by considering the control acting on $\boldsymbol{c}$ but not on the reference, and expressing $\boldsymbol{F}_p$ as function of the IC, Eq.11 can be expanded as

$$\boldsymbol{\delta}\dot{\boldsymbol{c}} = \frac{\partial\left(\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\chi}} \boldsymbol{F}_p(t, \boldsymbol{c})\right)}{\partial \boldsymbol{c}}\Big|_{\boldsymbol{c}=\boldsymbol{c}_{ref}} \boldsymbol{\delta c} + \boldsymbol{\Gamma}(t, \boldsymbol{c}_{ref})\boldsymbol{u}(t) + \mathscr{O}(\boldsymbol{\delta c}^2, \boldsymbol{u}\boldsymbol{\delta c}) \tag{23}$$

At the first-order, Eq.23 is a LTV system in the relative IC state $\boldsymbol{\delta c}$, which is governed by a linear dynamics, first-order approximation of the effect of $\boldsymbol{F}_p$ on the original IC state. As formulated in Eq.23, the first-order $\boldsymbol{F}_p$ effect can be compensated geometrically, as done in the optimal guidance and control of the satellite relative motion (Sec.V-B).

V. Case Studies

A. UAV Path Planning under Integrable Dynamics

This case study represents a simplified example of UAV path planning. The UAV is modeled as a point mass, m, with inertial position, $\boldsymbol{r} = (x, y, z)^T$, subjected to gravity, $-\boldsymbol{g}$, aerodynamic drag, $-\boldsymbol{A}\dot{\boldsymbol{r}}$, and constant wind force, $m\boldsymbol{W} = m(W_x, W_y, W_z)^T$. The control action is the thrust, $\boldsymbol{T}(t) = m(u_x(t), u_y(t), u_z(t))^T$, which in a real scenario is provided by the UAV rotors, and is coupled with the UAV attitude. In this simplified example, the attitude is neglected and the integrable translational UAV dynamics is expressed by the 2^{nd} order ODE: $\ddot{\boldsymbol{r}} = -\boldsymbol{g} - \boldsymbol{A}\dot{\boldsymbol{r}}/m + \boldsymbol{W} + \boldsymbol{T}(t)/m$. Integrating and considering an hovering reference condition, the IC state is defined as $\boldsymbol{c} = \left[x_0/l, y_0/l, z_0/l, \frac{m}{A_x l}(\dot{x}_0 - \frac{mW_x}{A_x}), \frac{m}{A_y l}(\dot{y}_0 - \frac{mW_y}{A_y}), \frac{m}{A_z l}(\dot{z}_0 - \frac{mW_z}{A_z})\right]^T$, where l is an adimensionalization factor. The closed form map is inherently defined through integration, and it is a linear function of the IC state, due to the linearity of the governing ODE. Subsequently, the ICV equation is formalized simply as $\dot{\boldsymbol{c}} = \boldsymbol{\Gamma}(t)\boldsymbol{T}(t)/m$. The defined path planning scenario consists in moving the UAV from an initial Cartesian state $[\boldsymbol{r}_0, \dot{\boldsymbol{r}}_0] = [(5m, 0, 10m), \boldsymbol{0}]^T$ to the target state $[\boldsymbol{r}_{goal}, \dot{\boldsymbol{r}}_{goal}] = [(-5m, 5m, 15m), \boldsymbol{0}]^T$ on the time interval $[t_0, t_f] = [0s, 10s]$. Other assigned values are: $m = 0.74kg$, $\boldsymbol{W} = [-0.1, 0.05, 0.05]m/s^2$ and $A_{x,y,z} = 0.15kg/s$. The dynamics in the Cartesian space is implemented in discrete sense by defining the state transition matrix (STM - $\boldsymbol{\Phi}(t, t_0)$) associated to the linear governing ODE. For this simple problem, Eq.15-16 and algorithm in Sec.IV-B can all be solved using a convex solver. They all lead to the same optimal solution and control profile demonstrating the consistency of the approach using IC. The minimum cost optimal control profile is a *bang-off-bang* solution composed by two impulses at time t_0 and t_f: $\Delta\boldsymbol{u}(t_0) = [-1.68, 0.84, 0.84]m/s$, $\Delta\boldsymbol{u}(t_f) = [0.65, -0.33, -0.33]m/s$. The advantage related to a definition of the path planning problem in IC space can be appreciated by looking at Fig.2. On the top, the optimal IC path (red line) is projected on the three IC sub-spaces, where it is superimposed to the reachable set $\mathscr{C}(J_{min})$ which is centered in $\boldsymbol{c}_0$ and includes $\boldsymbol{c}_{goal}$ on its border. $\mathscr{C}(J_{min})$ can be divided in two regions: the region reachable by a single impulse (colored), and the region reachable by $N \geq 1$ impulses (grey), the latter is the convex hull[13] of the former. $\boldsymbol{c}_{goal}$ is on the border of the grey region and, in particular, on the segment edge delimited by the sets reachable with an impulse at time t_0 ($\mathscr{C}(t_0, J_{min})$, yellow line) and t_f ($\mathscr{C}(t_f, J_{min})$, green line). Therefore, the optimal path must be a convex combination of points contained in these two delimiting sets leading to the optimal bang-off-bang profile. Consistently, the optimal IC path (red line) is a combination of a straight line into $\mathscr{C}(t_0, J_{min})$ and a line parallel to $\mathscr{C}(t_f, J_{min})$. The IC state moves only under action of control, therefore, the optimal path can be identified uniquely in a geometrical manner by tracing the segment parallel to $\mathscr{C}(t_f, J_{min})$ starting in $\mathscr{C}(t_0, J_{min})$ and ending into $\boldsymbol{c}_{goal}$. In Fig.2 on the bottom, the equivalent plots are showed into the Cartesian state space. Here, the set reachable with an impulse at time t_0 is the yellow region, corresponding to an initial velocity variation then mapped into position variation by the STM on the interval $[t_0, t_f]$. The reasoning made in IC space holds also in Cartesian space, but now the possibility to geometrically individuate the optimal path from $\boldsymbol{\chi}_0$ to $\boldsymbol{\chi}_{goal}$ is lost due to the action of the free integrable dynamics on the Cartesian state in between the two impulses, making not direct the deduction of the optimal path.

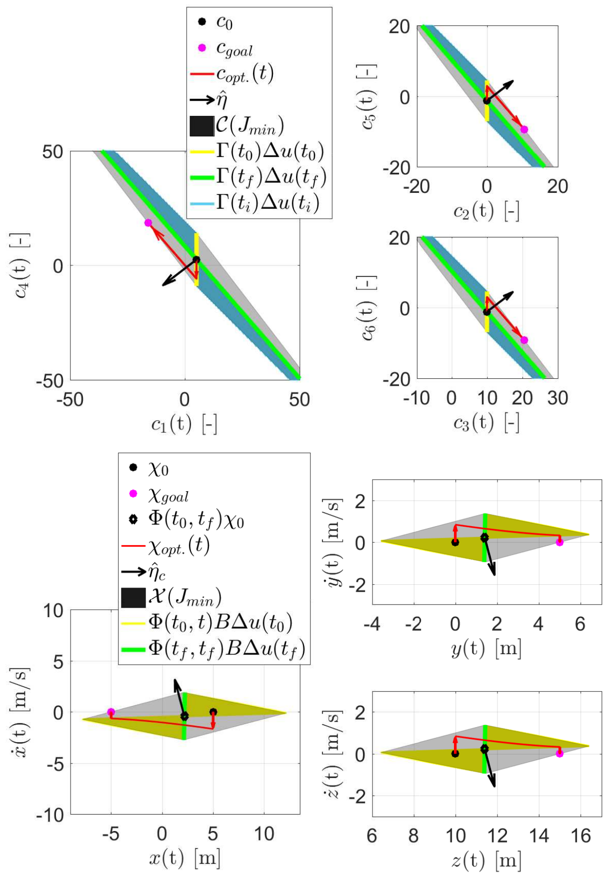

Fig. 2. UAV path planning - IC space (top), Cartesian space (bottom).

B. Satellite Relative Motion Control Under Non-Integrable Dynamics

The relative position between satellites in orbit is parameterized by means of relative orbital elements (ROE). The ROE as defined by D'Amico [2] are nonlinear combinations of the Keplerian orbital elements and IC of the homogeneous nonlinear equations of relative motion [16]. The effect of control and perturbations on the ROE is modeled by means of the Gauss Variational Equations [16][17][18], i.e., the ICV equation. Optimally control the satellite relative motion in ROE space permits to intrinsically leverage the integrable-Keplerian dynamics, and moreover, to include the effect of perturbations at the first-order and compensate them geometrically[5][16]. The geometrical approach to optimal control formalized in ROE space inspired the analysis presented in this paper and has been implemented on-board in two successful space missions [6][7].

VI. CONCLUSIONS

This paper shows how a state parameterization based on integration constants (IC) introduces a simplification in the optimal path planning of multi-dimensional systems governed by ODE. In particular, the solution of the optimal control problem in IC space leverages intrinsically the integrable part of the dynamics, while the perturbative part is isolated and then fully, or at least partially, compensated. This may open the way to a direct geometrical resolution of the optimal control problem. Future work of the authors will address further refinement of the presented approach through its application to more challenging scenarios.

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